

GLOBAL MODEL STRUCTURES FOR  $\ast$ -MODULES

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**ABSTRACT.** We extend Schwede's work on the unstable global homotopy theory of orthogonal spaces and  $\mathcal{L}$ -spaces to the category of  $\ast$ -modules (i.e., unstable  $S$ -modules). We prove a theorem which transports model structures and their properties from  $\mathcal{L}$ -spaces to  $\ast$ -modules and show that the resulting global model structure for  $\ast$ -modules is monoidally Quillen equivalent to that of orthogonal spaces. As a consequence, there are induced Quillen equivalences between the associated model categories of monoids, which identify equivalent models for the global homotopy theory of  $A_\infty$ -spaces.

## 1. INTRODUCTION

Global homotopy theory is equivariant homotopy theory with respect to compatible actions of the family of all compact Lie groups. Unstable global homotopy theory has been described by Schwede [12, 14] in terms of various model categories which make precise the idea of spaces with simultaneous group actions, studied up to *global equivalence*, i.e., weak  $G$ -homotopy equivalence with respect to all compact Lie groups  $G$ , in a compatible way.

Consider the following diagram of adjunctions (described in detail in Section 2):

$$(1.1) \quad \begin{array}{ccc} \mathcal{IU} & \xrightleftharpoons{Q} & \mathcal{LU} \\ & \searrow & \downarrow \uparrow F_{\boxtimes \mathcal{L}}(\ast, -) \\ & & \mathcal{M}_\ast \end{array}$$

The category  $\mathcal{IU}$  of *orthogonal spaces* is the category of diagram spaces indexed on the category  $\mathcal{I}$  of finite-dimensional real inner product spaces.

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It is symmetric monoidal under the Day convolution product. The global equivalences are the maps which, for each compact Lie group  $G$ , induce  $G$ -equivalences on homotopy colimits taken over all  $G$ -representations.

Let  $\mathcal{L}(1) = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$  be the space of linear isometric self-embeddings of an inner product space of countably infinite dimension. It is a topological monoid under composition. An  $\mathcal{L}$ -space is a space equipped with a continuous monoid action of  $\mathcal{L}(1)$ . We write  $\mathcal{LU}$  for the category of  $\mathcal{L}$ -spaces and  $\mathcal{L}(1)$ -equivariant maps. Any compact Lie group is isomorphic to a so-called *completely universal subgroup* of  $\mathcal{L}(1)$ . A map of  $\mathcal{L}$ -spaces is a global equivalence if it is a weak homotopy equivalence on  $G$ -fixed points for all completely universal subgroups  $G$ .

The respective global equivalences occur as weak equivalences of model structures on the categories of orthogonal spaces and  $\mathcal{L}$ -spaces, established by Schwede in [12] and [14]. There, it is also shown that the horizontal functor  $\mathbf{Q}$  depicted in Diagram 1.1 is a left Quillen equivalence with respect to these model structures.

The category of  $\mathcal{L}$ -spaces suffers from a small defect: It comes with a coherently associative and commutative *box product*  $-\boxtimes_{\mathcal{L}}-$ , which is only unital up to global equivalence. We refer to this structure as a *weak symmetric monoidal category*. The category  $\mathcal{M}_*$  of  $*$ -modules is the full subcategory of  $\mathcal{L}$ -spaces such that the unital transformations are isomorphisms. It is symmetric monoidal in the usual sense.

The model structures just mentioned are compatible with the respective (weak) symmetric monoidal products: They are monoidal model categories in the sense of Hovey [9, Def. 4.2.6]. It is natural to ask whether the model structure on  $\mathcal{L}$ -spaces can be transported to a Quillen equivalent monoidal model structure on  $*$ -modules along the vertical adjunction indicated in Diagram 1.1. The present work answers this question positively and thus provides a new framework for unstable global homotopy theory in the symmetric monoidal category of  $*$ -modules as an alternative to orthogonal spaces.

The first of the three main results presented here identifies easily checked conditions which are sufficient to transport “global” model structures and their properties from  $\mathcal{L}$ -spaces to  $*$ -modules. It was first proven in the author’s unpublished Master’s thesis [2]. The rigorous statement in its full strength and the proof are given in Section 5.

**Theorem A.** *Given a cofibrantly generated model structure on  $\mathcal{LU}$  with weak equivalences the global equivalences, and such that some mild conditions (specified*

in Theorem 5.1) are satisfied, there is a Quillen equivalent model structure on  $\mathcal{M}_*$  with weak equivalences the global equivalences and fibrations detected by the functor  $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{M}_* \rightarrow \mathcal{LU}$  (defined in Subsection 2.1). If the former is a monoidal model structure, then so is the latter.

The second main result is an immediate consequence: When applied to Schwede's global model structure  $(\mathcal{LU})_{\text{gl}}$  for  $\mathcal{L}$ -spaces, Theorem A yields the global model structure  $(\mathcal{M}_*)_{\text{gl}}$  for  $*$ -modules. Theorem B relates it to Schwede's positive global model structure  $(\mathcal{IU})_{\text{pos}}$  for orthogonal spaces via the horizontal Quillen equivalence  $Q$  mentioned before.

**Theorem B.** *There is a model structure  $(\mathcal{M}_*)_{\text{gl}}$  on  $*$ -modules with weak equivalences the global equivalences and other properties specified in Theorem 3.11. It fits into a triangle*

$$\begin{array}{ccc}
 (\mathcal{IU})_{\text{pos}} & \xrightleftharpoons{Q} & (\mathcal{LU})_{\text{gl}} \\
 & \searrow & \updownarrow F_{\boxtimes_{\mathcal{L}}}(*, -) \\
 & & (\mathcal{M}_*)_{\text{gl}}
 \end{array}$$

*of monoidal Quillen equivalences between monoidal model categories which commutes up to natural equivalence.*

The third main result states that Theorem B remains true for the associated categories of monoids, which carry model structures detected by the respective forgetful functors, as discussed in Section 4.

**Theorem C.** *The triangle from Theorem B lifts to a triangle of Quillen equivalences between the respective categories of monoids.*

Theorem C can be interpreted in terms of  $A_\infty$ -spaces: By a lemma of Hopkins, monoids in  $\mathcal{LU}$  are the same as algebras over the  $A_\infty$  operad  $\mathcal{L}$  of linear isometric embeddings, considered as a non-symmetric operad. Similarly, commutative monoids in  $\mathcal{LU}$  are the same as algebras over the  $E_\infty$  operad  $\mathcal{L}$ , considered as a symmetric operad. From this point of view, Theorem C states that monoids in orthogonal spaces and  $*$ -modules form equivalent models for the global homotopy theory of  $A_\infty$ -spaces. It is an open question whether the analogous result for  $E_\infty$ -spaces is true.

**Relation to other work:** Orthogonal spaces,  $\mathcal{L}$ -spaces and  $*$ -modules are the unstable counterparts of the category  $\mathcal{IS}$  of orthogonal spectra, the

category  $\mathcal{S}[\mathbb{L}]$  of  $\mathcal{L}$ -spectra, and the category  $\mathcal{M}_S$  of  $S$ -modules, respectively. They can be arranged as follows:

$$\begin{array}{ccccc}
 \mathcal{IS} & \xrightleftharpoons{\quad} & \mathcal{S}[\mathbb{L}] & \xrightleftharpoons{\quad} & \mathcal{M}_S \\
 \Omega^\infty \downarrow & & \Omega^\infty \downarrow & & \Omega^\infty \downarrow \\
 \mathcal{IU} & \xrightleftharpoons[\mathcal{Q}^\#]{\mathcal{Q}} & \mathcal{LU} & \xrightleftharpoons[\mathbb{F} \boxtimes_{\mathcal{L}}(*, -)]{(-) \boxtimes_{\mathcal{L}} *} & \mathcal{M}_*
 \end{array}$$

All of the categories carry non-equivariant model structures such that the horizontal arrows are Quillen equivalences and the infinite loop space functors  $\Omega^\infty$  are right Quillen functors. Upon passage to homotopy categories, the classical unstable and stable homotopy categories are obtained from any of the columns. See [10] for an exposition and further references. The proof of Theorem A is an adaptation of how the non-equivariant model structures are transported from  $\mathcal{L}$ -spectra to  $S$ -modules in [5] and from  $\mathcal{L}$ -spaces to  $*$ -modules in [1].

For a fixed group  $G$ , orthogonal spectra and  $S$ -modules with additional structure encoding the  $G$ -action have been studied equivariantly, see e.g. [11, 7, 13]. These additional data are not necessary in global homotopy theory. For each compact Lie group  $G$ , the  $G$ -equivariant homotopy groups of an ordinary orthogonal spectrum can be defined by evaluating only at  $G$ -representations. This idea gives rise to the global homotopy theory of orthogonal spectra and orthogonal spaces developed by Schwede in the pre-print [12].

Schwede's work includes variants that don't take into account all compact Lie groups, but only a certain family of groups. Hausmann [6] gave an equivalent description in the case of all finite groups, using symmetric spectra as a model.

**Organization:** We collect some preliminaries and known results about the categories and functors from Diagram 1.1 in Section 2. In Section 3, we describe Schwede's global model structure for  $\mathcal{L}$ -spaces and derive Theorem B. We show in Section 4 that these results carry over to monoids, thus proving Theorem C. Finally, the precise statement and proof of Theorem A is given in Section 5.

**Conventions:** We work over the category  $\mathcal{U}$  of compactly generated weak Hausdorff spaces. A *model category* is a Quillen model category as defined in [4, Def. 3.3]. The definition does not require functorial factorizations. A *monoidal model category* satisfies the pushout product axiom and the unit axiom, see [9, Def. 4.2.6]. An  *$h$ -cofibration* in a model category tensored

over  $\mathcal{U}$  is a map that satisfies the homotopy extension property. In diagrams, the upper or left arrow of an adjunction is the left adjoint.

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## 2. PRELIMINARIES

This section provides some background on the categories of  $\mathcal{L}$ -spaces,  $\ast$ -modules and orthogonal spaces, as well as on  $G$ -universes and universal subgroups. It does not contain any original results. The main sources are Schwede's preprint [14], his book project [12] and the article [1] by Blumberg, Cohen and Schlichtkrull.

**2.1.  $\mathcal{L}$ -spaces and  $\ast$ -modules.** Let  $\mathbf{L}(V, W)$  be the space of linear isometric embeddings  $V \rightarrow W$  between two real inner product spaces of finite or countable dimension, topologized as described in [10, Appendix A]. Write  $\mathbb{R}^\infty := \bigoplus_{\mathbb{N}} \mathbb{R}$  for the standard inner product space of countable dimension. The operad of linear isometric embeddings  $\mathcal{L}$  is given by spaces  $\mathcal{L}(n) = \mathbf{L}((\mathbb{R}^\infty)^n, \mathbb{R}^\infty)$ , and structure maps induced by direct sum and composition of maps. It is a (symmetric)  $E_\infty$ -operad with  $\Sigma_n$ -actions via permutation of the  $n$  summands of  $(\mathbb{R}^\infty)^n$ . The space of unary operations  $\mathcal{L}(1)$  is a topological monoid under composition, and we will study  $\mathcal{L}(1)$ -equivariant homotopy theory.

**Definition 2.1.** An  $\mathcal{L}$ -space is a space  $X \in \mathcal{U}$  together with a continuous action from the monoid  $\mathcal{L}(1)$ . We write  $\mathcal{LU}$  for the category of  $\mathcal{L}$ -spaces and  $\mathcal{L}(1)$ -equivariant maps.

The category  $\mathcal{LU}$  is bicomplete where (co-)limits are taken in the category  $\mathcal{U}$  of spaces and equipped with the respective (co-)limit action, because the forgetful functor to spaces has both adjoints. Moreover,  $\mathcal{LU}$  is enriched, tensored and co-tensored over  $\mathcal{U}$ .

The *box product* of  $\mathcal{L}$ -spaces  $X$  and  $Y$  is the balanced product

$$X \boxtimes_{\mathcal{L}} Y := \mathcal{L}(2) \times_{\mathcal{L}(1)^2} (X \times Y).$$

**Lemma 2.2** (Hopkins' lemma, see [5], Lemma I.5.4). *For  $m, n \geq 1$ , the space  $\mathcal{L}(m+n)$  is a split coequalizer of the diagram*

$$\mathcal{L}(2) \times \mathcal{L}(1)^2 \times (\mathcal{L}(m) \times \mathcal{L}(n)) \rightrightarrows \mathcal{L}(2) \times (\mathcal{L}(m) \times \mathcal{L}(n)),$$

hence  $\mathcal{L}(m) \boxtimes_{\mathcal{L}} \mathcal{L}(n) \cong \mathcal{L}(m+n)$  as  $\mathcal{L}$ -spaces.

The box product admits coherent associativity and commutativity isomorphisms and a right adjoint  $F_{\boxtimes_{\mathcal{L}}}(Y, -)$  for the functor  $- \boxtimes_{\mathcal{L}} Y: \mathcal{LU} \rightarrow \mathcal{LU}$ . See [1, Sect. 4.1] and [2, Def. 2.19]; cf. also [5, Sect. I.5]. It also admits a natural transformation

$$\begin{aligned} \lambda_{X,Y}: \quad X \boxtimes_{\mathcal{L}} Y = \mathcal{L}(2) \times_{\mathcal{L}(1)^2} (X \times Y) &\rightarrow X \times Y \\ [\psi_1 \oplus \psi_2, (x, y)] &\mapsto (\psi_1 \cdot x, \psi_2 \cdot y). \end{aligned}$$

which restricts to a *unital transformation*

$$\begin{aligned} \lambda_X: \quad X \boxtimes_{\mathcal{L}} * = \mathcal{L}(2) \times_{\mathcal{L}(1)^2} (X \times *) &\rightarrow X \\ [\psi_1 \oplus \psi_2, (x, *)] &\mapsto \psi_1 \cdot x. \end{aligned}$$

Here we used that each linear isometric embedding  $\psi: \mathbb{R}^\infty \oplus \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$  is given as  $\psi_1 \oplus \psi_2$ , where the  $\psi_i \in \mathcal{L}(1)$  have orthogonal images. Unfortunately,  $\lambda$  fails to be an isomorphism for all  $\mathcal{L}$ -spaces: For instance, all linear maps in the image of  $\lambda_{\mathcal{L}(1)}$  have an infinite-dimensional orthogonal complement, hence it is not surjective.

However,  $\lambda_X$  is always a weak equivalence of underlying spaces, see [1, Sect. 4.1], and it satisfies an even stronger, equivariant notion of equivalence, see Proposition 3.2. In order to be able to refer to this situation, we make the following definition.

**Definition 2.3.** A relative category  $(\mathcal{C}, \mathcal{W})$  is called a *weak (closed) symmetric monoidal category* if it is (closed) symmetric monoidal in the usual sense except that the left and right unital transformations are only required to lie in the class of weak equivalences  $\mathcal{W}$ , not necessarily in the class of isomorphisms.

**Definition 2.4.** A *\*-module* is an  $\mathcal{L}$ -space  $M$  such that  $\lambda_M$  is an isomorphism of  $\mathcal{L}$ -spaces.

Surprisingly, the quotient  $* \boxtimes_{\mathcal{L}} * = \mathcal{L}(2)/\mathcal{L}(1)^2$  is trivial, as proven in [5, Lemma I.8.1]. Consequently, the functor  $- \boxtimes_{\mathcal{L}} *$  on  $\mathcal{L}$ -spaces takes values in  $*$ -modules, and the box product restricts to a well-defined product on  $\mathcal{M}_*$ , which we denote by the same symbol  $\boxtimes_{\mathcal{L}}$ . So the category  $\mathcal{LU}$  is a

weak closed symmetric monoidal category, and then  $\mathcal{M}_\ast$  is a symmetric monoidal category in the usual sense. The latter is also closed, as follows formally from Proposition 2.5 below.

Dually, we let  $\mathcal{M}^\ast$  be the full subcategory of those  $\mathcal{L}$ -spaces such that the adjoint  $\bar{\lambda}_Y: Y \rightarrow F_{\boxtimes \mathcal{L}}(\ast, Y)$  is an isomorphism, and refer to its objects as *co-unital  $\mathcal{L}$ -spaces* or *co- $\ast$ -modules*. The functor  $F_{\boxtimes \mathcal{L}}(\ast, -)$  on  $\mathcal{LU}$  takes values in  $\mathcal{M}^\ast$ .

The following collection of statements from [1, Sect. 4.3] is an easy exercise in elementary category theory. It is the unstable analogue of a similar “mirror image” argument for  $S$ -modules, cf. [5, Sect. II.2].

**Proposition 2.5.** *The categories  $\mathcal{M}_\ast$  and  $\mathcal{M}^\ast$  of unital and co-unital  $\mathcal{L}$ -spaces, respectively, are “mirror image subcategories” in the following sense:*

- a) *All pairs of functors in the diagram below form adjunctions (where upper arrows and arrows on the left hand side are left adjoints).*

$$\begin{array}{ccc}
 \mathcal{LU} & \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}} \ast} \\ \xleftarrow{F_{\boxtimes \mathcal{L}}(\ast, -)} \end{array} & \mathcal{M}_\ast \\
 \begin{array}{c} \uparrow \text{ } F_{\boxtimes \mathcal{L}}(\ast, -) \\ \downarrow \text{ } \end{array} & \begin{array}{c} \swarrow -\boxtimes_{\mathcal{L}} \ast \\ \searrow F_{\boxtimes \mathcal{L}}(\ast, -) \end{array} & \begin{array}{c} \uparrow \text{ } -\boxtimes_{\mathcal{L}} \ast \\ \downarrow \text{ } \end{array} \\
 \mathcal{M}^\ast & \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}} \ast} \\ \xleftarrow{F_{\boxtimes \mathcal{L}}(\ast, -)} \end{array} & \mathcal{LU}
 \end{array}$$

- b) *The subdiagrams of left-adjoint (respectively right-adjoint) functors commute up to natural equivalence.*
- c) *The categories  $\mathcal{M}_\ast$  and  $\mathcal{M}^\ast$  are bicomplete. Colimits in  $\mathcal{M}_\ast$  are created in  $\mathcal{LU}$ , limits are obtained by applying  $-\boxtimes_{\mathcal{L}} \ast$  to limits in  $\mathcal{LU}$ ; dually for  $\mathcal{M}^\ast$ .*
- d) *The diagonal adjunction (co-)restricts to an equivalence of categories*

$$\mathcal{M}^\ast \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}} \ast} \\ \xleftarrow{F_{\boxtimes \mathcal{L}}(\ast, -)} \end{array} \mathcal{M}_\ast.$$

**2.2. Universal subgroups.** In this short section, we recall that every compact Lie group is isomorphic to an actual subgroup of  $\mathcal{L}(1)$ , a so-called *universal subgroup*. Thus, all compact Lie groups act simultaneously on each  $\mathcal{L}$ -space  $X$ ; these actions are compatible in the sense that they all are restrictions of the same action of  $\mathcal{L}(1)$  on  $X$ .

**Definition 2.6.** Let  $U_G$  be an orthogonal  $G$ -representation of countable dimension. We say that  $U_G$  is



- i) a *G-preuniverse* if for each finite-dimensional  $G$ -representation  $V$  that embeds into  $U_G$ , the representation  $\bigoplus_{\mathbb{N}} V$  also embeds into  $U_G$ ,
- ii) a *G-universe* if it is a preuniverse that contains a 1-dimensional (hence a countable dimensional) trivial subrepresentation,
- iii) a *faithful G-(pre-)universe* if it is a  $G$ -(pre-)universe that is faithful when considered as a  $G$ -representation,
- iv) a *complete G-universe* if it is a  $G$ -universe that contains one copy, and hence countably many copies of each irreducible  $G$ -representation.

**Definition 2.7** ([14], Def. 2.10). A compact subgroup  $G \leq \mathcal{L}(1)$  is called *universal* (respectively *completely universal*) if it admits the structure of a compact Lie group (necessarily unique, see [3, Prop. 3.12]) such that under the tautological action,  $\mathbb{R}^\infty$  becomes a  $G$ -preuniverse (respectively a complete  $G$ -universe).

Let  $G \leq \mathcal{L}(1)$  be a universal subgroup, then we write  $\mathbb{R}_G^\infty$  for the resulting faithful preuniverse under the tautological action on  $\mathbb{R}^\infty$ .

**Lemma 2.8** (cf. [14], Prop. 2.11). *The equivalence classes of completely universal subgroups of  $\mathcal{L}(1)$  under conjugation by invertible elements of  $\mathcal{L}(1)$  biject with the isomorphism classes of compact Lie groups.*

In Section 3, we will introduce various notions of equivalences and fibrations detected on  $G$ -fixed points for all (completely) universal subgroups  $G \leq \mathcal{L}(1)$ .

**2.3. Orthogonal spaces.** Write  $\mathcal{I}$  for the category of finite-dimensional real inner product spaces with morphisms the linear isometric embeddings. It is enriched over spaces, see the beginning of Subsection 2.1.

**Definition 2.9.** An *orthogonal space* is a continuous functor  $Y: \mathcal{I} \rightarrow \mathcal{U}$ . We write  $\mathcal{IU}$  for the category of orthogonal spaces and natural transformations.

The category  $\mathcal{IU}$  is bicomplete, with (co-)limits taken objectwise. Moreover, it is tensored and co-tensored over  $\mathcal{U}$  where, for  $Y \in \mathcal{IU}$ ,  $A \in \mathcal{U}$ , the tensor space  $Y \times A$  sends  $V \in \mathcal{I}$  to  $(Y \times A)(V) := Y(V) \times A$ . Equivalently, we can regard  $A$  as the constant orthogonal space with value  $A$  and form the product in  $\mathcal{IU}$ .

The category of orthogonal spaces is a closed symmetric monoidal category under the *box product*, which is the Day convolution product with respect to direct sum of vector spaces in  $\mathcal{I}$  and the product in  $\mathcal{U}$ , see [12,



Sect. I.5, A.3] for further details. A unit is given by the constant one-point orthogonal space  $\mathbb{1}$ .

Following Schwede, we take *global equivalences* of orthogonal spaces to be those morphisms that, for each compact Lie group  $G$ , induce  $G$ -weak equivalences on homotopy colimits along  $G$ -representations. The precise definition is given in more elementary terms, cf. [12, Rem. I.1.4].

**Definition 2.10** ([12], Def. I.1.2). A morphism  $f: X \rightarrow Y$  of orthogonal spaces is a *global equivalence* if for any compact Lie group  $G$ , any orthogonal  $G$ -representation  $V$  of finite dimension, any  $k \geq 0$  and any commuting square

$$\begin{array}{ccc} S^{k-1} & \xrightarrow{\alpha} & X(V)^G \\ \text{incl} \downarrow & & \downarrow f(V)^G \\ D^k & \xrightarrow{\beta} & Y(V)^G \end{array}$$

there is a finite-dimensional  $G$ -representation  $W$ , a  $G$ -equivariant linear isometric embedding  $\varphi: V \rightarrow W$  and a map  $\lambda: D^k \rightarrow X(W)^G$  such that in the extended diagram

$$\begin{array}{ccccc} S^{k-1} & \xrightarrow{\alpha} & X(V)^G & \xrightarrow{X(\varphi)^G} & X(W)^G \\ \text{incl} \downarrow & & & \nearrow \lambda & \downarrow f(W)^G \\ D^k & \xrightarrow{\beta} & Y(V)^G & \xrightarrow{Y(\varphi)^G} & Y(W)^G \end{array}$$

the upper triangle commutes strictly and the lower triangle commutes up to homotopy relative to  $S^{k-1}$ .

**Theorem 2.11** (Global model structures for orthogonal spaces, cf. [12], Thm. I.4.10, Prop. II.3.3). *The global equivalences are part of two proper, topological, cofibrantly generated model structures on the category of orthogonal spaces, the (absolute) global model structure  $(\mathcal{IU})_{\text{abs}}$  and the positive global model structure  $(\mathcal{IU})_{\text{pos}}$ .*

We omit the description of the (co-)fibrations in the two global model structures since we will not make use of them explicitly. As usual, the positive variant has a better behaviour with respect to commutative monoids, and for different reasons, it is also necessary for us to work with the positive global model structure throughout the paper, see Remark 3.10. The absolute model structure will only appear in Section 4.

Note that both global model structures are monoidal: The pushout product axiom is proven in [12, Prop. I.5.7, Prop. II.3.3], while the unit axiom follows from [12, Thm. I.5.3].

The categories  $\mathcal{IU}$  and  $\mathcal{LU}$  can be connected by an adjoint pair of functors. By general theory, the right exact enriched functors (i.e., those which preserve colimits and tensors)  $\mathcal{DU} \rightarrow \mathcal{C}$  out of a category of diagram spaces into a cocomplete category  $\mathcal{C}$  agree with the enriched functors  $\mathcal{D}^{op} \rightarrow \mathcal{C}$  up to isomorphism of categories; see [11, Sect. I.2] and note that the results also apply in the unbased case.

Lind [10, Def. 8.2] defines a functor  $Q^*: \mathcal{I}^{op} \rightarrow \mathcal{LU}$  that sends  $V$  to  $\mathcal{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty)$ ; it is strong symmetric monoidal by Lemma 2.2. The results of [11] then yield an adjunction

$$\mathcal{IU} \begin{array}{c} \xrightarrow{Q} \\ \xleftarrow{Q^\#} \end{array} \mathcal{LU}.$$

where the left adjoint  $Q$  is given as an enriched coend  $Q^* \otimes_{\mathcal{I}} (-)$  and the right adjoint is  $Q^\# X(V) = \mathcal{LU}(Q^*(V), X)$ . The first is strong, the latter lax symmetric monoidal.

The functor  $Q^*: \mathcal{I} \rightarrow \mathcal{LU}$  can be replaced by a functor  $Q_*^*$  defined as  $V \mapsto \mathcal{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \boxtimes_{\mathcal{L}} *$ . It takes values in  $*$ -modules and yields an adjunction

$$\mathcal{IU} \begin{array}{c} \xrightarrow{Q_*} \\ \xleftarrow{Q_*^\#} \end{array} \mathcal{M}_*.$$

defined in the same way as before. Again, the left adjoint is strong, the right adjoint lax symmetric monoidal. By [10, Lemma 8.6], this pair of functors agrees, up to natural equivalence, with the composition of adjunctions

$$\mathcal{IU} \begin{array}{c} \xrightarrow{Q} \\ \xleftarrow{Q^\#} \end{array} \mathcal{LU} \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}} *} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*, -)} \end{array} \mathcal{M}_*.$$

*Remark 2.12.* There is another interesting choice of a functor  $\mathcal{IU} \rightarrow \mathcal{LU}$ . For an orthogonal space  $Y$ , the colimit  $Y(\mathbb{R}^\infty) := \text{colim}_V Y(V)$  taken over all finite-dimensional inner product subspaces  $V \subseteq \mathbb{R}^\infty$  (or equivalently, all standard Euclidean spaces  $\mathbb{R}^n$ ) has a canonical  $\mathcal{L}$ -space structure, see [14, Ex. 2.4]. The resulting functor  $\mathcal{IU} \rightarrow \mathcal{LU}$  is induced by  $O^*: \mathcal{I} \rightarrow \mathcal{LU}$  sending  $V \in \mathcal{I}$  to  $L(V, \mathbb{R}^\infty)$ , see [10, Lemma 9.6]. Schwede proved that  $O$  is strong symmetric monoidal, see [14, Prop. 4.5]. It follows formally that its right adjoint is a lax symmetric monoidal functor.

Any choice of a one-dimensional subspace of  $\mathbb{R}^\infty$  defines a linear isometric embedding  $V \rightarrow V \otimes \mathbb{R}^\infty$ , hence a natural transformation  $\tilde{\zeta}^*: \mathcal{Q}^*(V) = \mathcal{L}(V \otimes \mathbb{R}^\infty, \mathbb{R}^\infty) \rightarrow \mathcal{L}(V, \mathbb{R}^\infty) \cong \mathcal{O}^*(V)$  which in turn determines a natural map  $\tilde{\zeta} = \tilde{\zeta}^* \otimes_I (-): \mathcal{Q} \rightarrow \mathcal{O}$ . The latter is symmetric monoidal; moreover, it is a global equivalence on cofibrant objects in the absolute model structure on orthogonal spaces, see [14, Prop. 3.9]. A precursor of the last statement was [10, Lemma 9.7].

### 3. MODEL STRUCTURES FOR $\mathcal{LU}$ AND $\mathcal{M}_\ast$

We recall Schwede's model structures for  $\mathcal{L}$ -spaces from [14] and derive Theorem B. It establishes the global model structure for  $\ast$ -modules which is Quillen equivalent to orthogonal spaces via the functor  $\mathcal{Q}$ .

The following notions of equivalences and fibrations of  $\mathcal{L}$ -spaces will also be used for maps of  $\ast$ -modules by viewing them as maps in  $\mathcal{LU}$ .

**Definition 3.1.** A map  $f: X \rightarrow Y$  of  $\mathcal{L}$ -spaces is called

- a *universal equivalence* (respectively *universal fibration*) if the map  $f^G: X^G \rightarrow Y^G$  is a weak homotopy equivalence (respectively Serre fibration) of spaces for all universal subgroups  $G \leq \mathcal{L}(1)$ ;
- a *global equivalence* if  $f^G: X^G \rightarrow Y^G$  is a weak homotopy equivalence of spaces for all completely universal subgroups  $G \leq \mathcal{L}(1)$ ;
- a *strong universal equivalence* (respectively *strong global equivalence*) if  $f$ , considered as a map of  $G$ -spaces, is a  $G$ -equivariant homotopy equivalence for all (completely) universal subgroups  $G \leq \mathcal{L}(1)$ .

**Proposition 3.2.** *The natural map of  $\mathcal{L}$ -spaces  $\lambda_{X,Y}: X \boxtimes_{\mathcal{L}} Y \rightarrow X \times Y$  is a strong global equivalence for all  $X, Y \in \mathcal{LU}$ . Consequently, so is the adjoint map  $\bar{\lambda}_X: X \rightarrow F_{\boxtimes_{\mathcal{L}}}(\ast, X)$ . Both functors  $(-) \boxtimes_{\mathcal{L}} \ast$  and  $F_{\boxtimes_{\mathcal{L}}}(\ast, -)$  preserve and reflect (strong) global equivalences. For all  $Z \in \mathcal{LU}$ , the functor  $(-) \boxtimes_{\mathcal{L}} Z$  preserves (strong) global equivalences.*

*Proof.* The first part is [14, Thm. 4.2]. In combination with the 2-out-of-3 property and the following diagram, it implies the second statement; the third then follows immediately.

$$\begin{array}{ccc}
 X \boxtimes_{\mathcal{L}} \ast & \xrightarrow[\sim]{\lambda_X} & X \\
 \bar{\lambda}_X \boxtimes_{\mathcal{L}} \ast \downarrow \cong & & \downarrow \bar{\lambda}_X \\
 F_{\boxtimes_{\mathcal{L}}}(\ast, X) \boxtimes_{\mathcal{L}} \ast & \xrightarrow[\sim]{\lambda_{F_{\boxtimes_{\mathcal{L}}}(\ast, X)}} & F_{\boxtimes_{\mathcal{L}}}(\ast, X)
 \end{array}$$

Now let  $Z \in \mathcal{LU}$  be arbitrary. If  $f$  is a (strong) global equivalence, then so is  $f \times Z$ , hence also  $f \boxtimes_{\mathcal{L}} Z$ .  $\square$

*Remark 3.3.* The statements of Proposition 3.2 remain true for (strong) universal equivalences since all the steps in the proof of [14, Thm. 4.2] only require faithful (but not necessarily complete) preuniverses.

Recall that for any  $G \leq \mathcal{L}(1)$ , the  $\mathcal{L}$ -space  $\mathcal{L}(1)/G$  represents the fixed point functor  $(-)^G: \mathcal{LU} \rightarrow \mathcal{U}$ . For any category of spaces with monoid action, so-called “bi-closed” collections of submonoids yield model structures with equivalences and fibrations detected on the corresponding collections of fixed points, see [14, Prop. 1.10] for details. The next statement is a special case.

**Proposition 3.4** (Universal model structure for  $\mathcal{L}$ -spaces, [14], Prop. 2.15). *There is a proper topological model structure  $(\mathcal{LU})_{\text{univ}}$  on the category of  $\mathcal{L}$ -spaces with weak equivalences and fibrations the universal equivalences and universal fibrations. It is cofibrantly generated with sets of generating (acyclic) cofibrations obtained by tensoring the standard generating (acyclic) cofibrations for spaces  $S^{n-1} \rightarrow D^n$  (respectively  $D^n \times 0 \rightarrow D^n \times I$ ) with  $\mathcal{L}$ -spaces of the form  $\mathcal{L}(1)/G$ , where  $G$  runs through all universal subgroups of  $\mathcal{L}(1)$ .*

The universal model structure seems unlikely to be Quillen equivalent to  $\mathcal{IU}$  with its positive global model structure. Following Schwede’s approach, we apply a left Bousfield localization such that the weak equivalences become precisely the class of global equivalences. This detour is necessary in order to guarantee that the adjunction to orthogonal spaces becomes a Quillen adjunction.

Given two universal subgroups  $G, \bar{G} \leq \mathcal{L}(1)$  and an isomorphism of Lie groups  $\alpha: G \rightarrow \bar{G}$ , then for any  $G$ -equivariant linear isometric embedding  $\varphi: \alpha^*(\mathbb{R}_G^\infty) \rightarrow \mathbb{R}_{\bar{G}}^\infty$  and any  $X \in \mathcal{LU}$ , the map

$$\varphi \cdot (-): X \rightarrow X, x \mapsto \varphi \cdot x$$

satisfies  $g \cdot (\varphi \cdot x) = \varphi \cdot (\alpha(g) \cdot x)$ .

**Definition 3.5** ([14], Def. 2.16). An  $\mathcal{L}$ -space  $X$  is called *injective* if the induced map of spaces  $\varphi \cdot (-): X^{\bar{G}} \rightarrow X^G$  is a weak homotopy equivalence for all choices of  $G, \bar{G}, \alpha$  and  $\varphi$ .

The natural maps  $\varphi \cdot (-): X^{\bar{G}} \rightarrow X^G$  are represented by  $\mathcal{L}$ -maps

$$\varphi_\#: \mathcal{L}(1)/G \rightarrow \mathcal{L}(1)/\bar{G}, \theta \cdot G \mapsto (\theta\varphi) \cdot \bar{G}.$$

Now a left Bousfield localization at the set of all morphisms  $\varphi_{\#}$  yields the desired model category. Its existence is not an issue because  $(\mathcal{LU})_{\text{univ}}$  can be shown to be cellular in the sense of [8, Def. 12.1.1]. The part of the proof which is not obvious proceeds as in [8, 10.7.4]; compare also [2, Prop. 4.11].

**Theorem 3.6** (Global model structure for  $\mathcal{L}$ -spaces, see [14], Thm. 2.24, Prop. 4.1). *There is a cofibrantly generated proper topological model structure  $(\mathcal{LU})_{\text{gl}}$  on the category of  $\mathcal{L}$ -spaces with weak equivalences the global equivalences and cofibrations as in  $(\mathcal{LU})_{\text{univ}}$ . The fibrant objects are the injective  $\mathcal{L}$ -spaces.*

**Proposition 3.7.** *The model structure  $(\mathcal{LU})_{\text{univ}}$  is a monoidal model category.*

*Proof.* The pushout product axiom is proven in [14, Prop. 4.1], the unit axiom follows from Proposition 3.2.  $\square$

Schwede also proved that the homotopy categories of his global model structures for orthogonal spaces and  $\mathcal{L}$ -spaces agree, as we record now.

**Theorem 3.8** ([14], Thm. 3.7). *The adjunction*

$$(\mathcal{IU})_{\text{pos}} \begin{array}{c} \xrightarrow{\mathcal{Q}} \\ \xleftarrow{\quad} \end{array} (\mathcal{LU})_{\text{gl}}$$

*is a Quillen equivalence.*

**Remark 3.9.** The proof given in [14] is an equivariant adaptation of the proof of [10, Thm. 9.9]. We point out for later reference that it even shows the following: If  $C \in \mathcal{IU}$  is only cofibrant in the absolute model structure and  $X \in \mathcal{LU}$  is injective, then a map  $\mathcal{Q}C \rightarrow X$  is a global equivalence if and only if its adjoint  $C \rightarrow \mathcal{Q}^{\#}X$  is.

**Remark 3.10.** The functor  $\mathcal{Q}^{\#}$  is not a right Quillen functor anymore if we use the absolute model structure on orthogonal spaces instead: If  $\mathcal{Q}^{\#}X$  is fibrant in the absolute model structure, then the inclusion of fixed points  $X^{\mathcal{L}(1)} \rightarrow X$  must be a weak homotopy equivalence of spaces by [12, Def. I.4.1], which cannot be true for each injective  $\mathcal{L}$ -space  $X$ .

Assuming Theorem 5.1 (i.e., Theorem A), we will now prove the following:

**Theorem 3.11** (Theorem B). *There is a cofibrantly generated proper topological model structure on the category  $\mathcal{M}_{*}$  of  $*$ -modules, the global model structure  $(\mathcal{M}_{*})_{\text{gl}}$ . Its weak equivalences are the global equivalences of underlying  $\mathcal{L}$ -spaces, its fibrations are detected by the functor  $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{M}_{*} \rightarrow (\mathcal{LU})_{\text{gl}}$ . Let  $I$  and  $J$  be any sets of generating (acyclic) cofibrations for  $(\mathcal{LU})_{\text{gl}}$ , then  $I \boxtimes_{\mathcal{L}} *$  and  $J \boxtimes_{\mathcal{L}} *$  are generating (acyclic) cofibrations for  $(\mathcal{M}_{*})_{\text{gl}}$ .*

Moreover, the global model structure for  $\mathcal{M}_*$  is monoidal and satisfies the monoid axiom [15, Def. 3.3] with respect to  $\boxtimes_{\mathcal{L}}$ . It fits into the following commutative (up to natural isomorphism) triangle of monoidal Quillen equivalences:

$$\begin{array}{ccc}
 (\mathcal{U})_{\text{pos}} & \xrightleftharpoons{\mathcal{Q}} & (\mathcal{L}\mathcal{U})_{\text{gl}} \\
 & \searrow \mathcal{Q}_* & \downarrow \text{F}_{\boxtimes_{\mathcal{L}}}(*, -) \\
 & & (\mathcal{M}_*)_{\text{gl}}
 \end{array}$$

*Proof.* The global model structure obviously satisfies the requirements of Theorem A (Theorem 5.1), which immediately implies the existence and properties of the model structure  $(\mathcal{M}_*)_{\text{gl}}$ , except for the unit axiom, which follows from Proposition 3.2. It also proves that the vertical adjunction is a Quillen equivalence. The horizontal adjunction is a Quillen equivalence by the theorem above, and we have already seen that all adjunctions are monoidal.  $\square$

*Remark 3.12.* There is a variant of Theorem B with respect to the functor  $\mathcal{O}: \mathcal{U} \rightarrow \mathcal{L}\mathcal{U}$  introduced in Remark 2.12: It is possible to establish a model structure on  $\mathcal{L}$ -spaces with weak equivalences the global equivalences and such that  $\mathcal{O}$  is a left Quillen equivalence with respect to the absolute global model structure on orthogonal spaces. This model structure also satisfies the hypotheses of Theorem A, but is harder to work with as the cofibrations and fibrations cannot only be characterized in terms of fixed points of group actions. It also lifts to monoids and the analogue of Theorem C holds. Details can be found in the author's (unpublished) Master's thesis [2].

*Remark 3.13.* The diagram in Theorem B (Theorem 3.11) can be extended to the right: By a version of Elmendorf's theorem,  $(\mathcal{L}\mathcal{U})_{\text{gl}}$  is Quillen equivalent to a model category of "systems of global fixed point sets". As usual, these are diagram spaces indexed on the opposite of a suitable "global" orbit category. We refer to [14, Constr. 1.13, Prop. 1.14] for details.

#### 4. MONOIDS AND MODULES IN GLOBAL HOMOTOPY THEORY

Monoids with respect to  $\boxtimes_{\mathcal{L}}$  and their modules have been described non-equivariantly by Blumberg, Cohen and Schlichtkrull, see [1, Thm. 4.18]. We describe "global" analogues of their result and prove Theorem C.

Recall from Section 2 that  $\mathcal{L}$  denotes the operad of linear isometric embeddings of  $\mathbb{R}^\infty$ . The following identifications are a consequence of Hopkins' Lemma 2.2.

**Proposition 4.1** ([1], Prop. 4.7). *The category of  $A_\infty$ -spaces structured by  $\mathcal{L}$  (considered as a non-symmetric operad) is isomorphic to the category of  $\boxtimes_{\mathcal{L}}$ -monoids in  $\mathcal{LU}$ . The category of  $E_\infty$ -spaces structured by  $\mathcal{L}$  (considered as a symmetric operad) is isomorphic to the category of commutative  $\boxtimes_{\mathcal{L}}$ -monoids in  $\mathcal{LU}$ .*

**Corollary 4.2** ([1], Sect. 4.4). *The  $\boxtimes_{\mathcal{L}}$ -monoids in  $\mathcal{M}_*$  are those  $A_\infty$ -spaces which are  $*$ -modules. The functor  $-\boxtimes_{\mathcal{L}}*: \mathcal{LU} \rightarrow \mathcal{M}_*$  takes  $\boxtimes_{\mathcal{L}}$ -monoids in  $\mathcal{LU}$  to  $\boxtimes_{\mathcal{L}}$ -monoids in  $\mathcal{M}_*$  and the natural map  $\lambda_X: X \boxtimes_{\mathcal{L}} * \rightarrow X$  is a map of  $\boxtimes_{\mathcal{L}}$ -monoids if  $X$  is a  $\boxtimes_{\mathcal{L}}$ -monoid. The analogous statement is true for commutative monoids and  $E_\infty$ -spaces.*

In [15], Schwede and Shipley describe sufficient conditions for a cofibrantly generated monoidal model structure to lift to the associated categories of  $R$ -modules and  $R$ -algebras, respectively, where  $R$  is any (commutative) monoid. When applied to the global model structure on  $*$ -modules, this yields:

**Theorem 4.3.** *Consider the category of  $*$ -modules equipped with the global model structure and let  $R$  be a  $\boxtimes_{\mathcal{L}}$ -monoid in  $\mathcal{M}_*$ . Call a morphism of  $R$ -algebras a weak equivalence (respectively fibration) if it is a weak equivalence (respectively fibration) of underlying  $*$ -modules. With respect to these classes of morphisms, the following holds:*

- 1) *The category of left  $R$ -modules is a cofibrantly generated model category.*
- 2) *If  $R$  is commutative, then the category of  $R$ -modules is a cofibrantly generated monoidal model category satisfying the monoid axiom.*
- 3) *If  $R$  is commutative, then the category of  $R$ -algebras is a cofibrantly generated model category. If the source of a cofibration of  $R$ -algebras is cofibrant as an  $R$ -module, then the map is a cofibration of  $R$ -modules.*

*In all cases, sets of generating cofibrations and acyclic cofibrations are given by the images of generating sets for  $\mathcal{M}_*$  under the free functor.*

For  $R = *$ , the category of  $R$ -algebras is the category of  $\boxtimes_{\mathcal{L}}$ -monoids. It has a cofibrantly generated model structure by part 3) of the theorem.

*Proof.* We check the hypotheses of [15, Thm. 4.1]. As explained in [15, Rem. 4.2], the smallness assumption can be weakened; it then follows from the fact that the forgetful functors from  $R$ -modules and monoids,



respectively, commute with filtered colimits, and from Lemma 5.5.

By part h) of Theorem 5.1 (Theorem A),  $(\mathcal{M}_*)_{\text{gl}}$  satisfies the monoid axiom as defined in [15, Def. 3.3].  $\square$

**Theorem 4.4.** *The analogue of Theorem 4.3 with respect to the monoidal model category  $(\mathcal{LU})_{\text{gl}}$  is true.*

*Proof.* A close inspection of the proof of [15, Thm. 4.1] shows that the first two statements do not require that the unital transformation is an isomorphism, so these hold because  $(\mathcal{LU})_{\text{gl}}$  satisfies the monoid axiom, see part h) of Theorem 5.1. The proof of the third statement makes use of the unital isomorphism in order to verify that all relative  $J_T$ -cell complexes are weak equivalences. We will give an alternative proof of this fact instead:

Here,  $T: X \mapsto \coprod_{n \geq 0} X^{\boxtimes_{\mathcal{L}} n}$  is the composition of the free monoid functor with the forgetful functor,  $J$  is any set of generating acyclic cofibrations for  $(\mathcal{LU})_{\text{gl}}$ , and  $J_T$  denotes its image under  $T$ . All maps in  $J$  are  $h$ -cofibrations (i.e., have the homotopy extension property) and global equivalences. For each  $Z \in \mathcal{LU}$ , the left adjoint functor  $Z \boxtimes_{\mathcal{L}} (-): \mathcal{LU} \rightarrow \mathcal{LU}$  preserves these properties by Proposition 3.2 and Lemma 5.3. Thus, for a map  $j: A \rightarrow B$  in  $J$  and  $n \geq 2$ , we can write the  $n$ -th summand  $j^{\boxtimes_{\mathcal{L}} n}$  of  $T(j)$  as a composition

$$(j \boxtimes_{\mathcal{L}} A^{\boxtimes_{\mathcal{L}} (n-1)}) \circ (B \boxtimes_{\mathcal{L}} j \boxtimes_{\mathcal{L}} A^{\boxtimes_{\mathcal{L}} (n-2)}) \circ \dots \circ (B^{\boxtimes_{\mathcal{L}} (n-1)} \boxtimes_{\mathcal{L}} j).$$

of maps which are both  $h$ -cofibrations and global equivalences. These properties are stable under composition and coproducts, hence  $T(j)$  has both properties. Moreover, the class of  $h$ -cofibrations which are global equivalences is closed under cobase change and transfinite composition, thus each morphism in  $J_T$  is a global equivalence. Smallness is not an issue because all  $\mathcal{L}$ -spaces are small relative to closed embeddings (Lemma 5.5), and so relative to all images of cofibrations under  $T$ .  $\square$

**Theorem 4.5.** *The analogue of Theorem 4.3 with respect to the monoidal model categories  $(\mathcal{IU})_{\text{abs}}$  and  $(\mathcal{IU})_{\text{pos}}$  is true.*

*Proof.* Every acyclic cofibration in the positive global model structure on  $\mathcal{IU}$  is an acyclic cofibration in the absolute global model structure. The latter satisfies the monoid axiom, see [12, Prop. I.7.13], hence so does the former. Again, [15, Thm. 4.1] applies.  $\square$

**Theorem 4.6** (Theorem C). *The triangle of monoidal Quillen equivalences from Theorem B gives rise to a triangle of Quillen equivalences between the respective model structures on categories of monoids.*

*Proof of Theorem 4.6.* In all cases, the forgetful functors preserve and reflect fibrations and weak equivalences, thus the lifted right adjoints are always right Quillen functors. The induced adjunctions over  $\mathcal{LU}$  and  $\mathcal{M}_*$  are Quillen equivalences because the functor  $- \boxtimes_{\mathcal{L}} *$  preserves and reflects global equivalences and the counit  $F_{\boxtimes_{\mathcal{L}}}(*, X) \boxtimes_{\mathcal{L}} * \rightarrow X$  is an isomorphism for all  $X \in \mathcal{M}_*$ , see Proposition 2.5.

It remains to show that the Quillen adjunction between monoids in  $\mathcal{IU}$  and  $\mathcal{LU}$  is indeed a Quillen equivalence. Let  $M$  be a positively cofibrant monoid in  $\mathcal{IU}$  and  $N$  a fibrant monoid in  $\mathcal{LU}$ . Then  $N$  is an injective  $\mathcal{L}$ -space and  $M$  is absolutely cofibrant as a monoid. As the monoidal unit  $\mathbb{1} \in \mathcal{IU}$  is an absolutely cofibrant orthogonal space, the characterization of cofibrations in [15, Thm. 4.1] implies that  $M$  is absolutely cofibrant as an orthogonal space, too. The claim now follows immediately from the observation made in Remark 3.9.  $\square$

In light of Proposition 4.1, Theorem C states that there is an unambiguous global homotopy theory of  $A_\infty$ -spaces. We don't know if this statement is true for  $E_\infty$ -spaces: The positive global model structure  $(\mathcal{IU})_{\text{pos}}$  lifts to commutative monoids, see [12, Thm. II.3.9], but it remains open whether the same holds for  $(\mathcal{M}_*)_{\text{gl}}$ . The difficulty is in showing that the functor  $(-)^{\boxtimes_{\mathcal{L}}} / \Sigma_n$  takes acyclic cofibrations to global equivalences.

## 5. PROOF OF THEOREM A

We finally give a precise statement and proof of Theorem A. Throughout this section, let  $F$  denote the functor  $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{LU} \rightarrow \mathcal{M}^*$  and let  $R$  be its right adjoint, the forgetful functor  $\mathcal{M}^* \rightarrow \mathcal{LU}$ .

**Theorem 5.1** (Theorem A). *Let  $(\mathcal{LU})_a$  be any model structure on the category  $\mathcal{LU}$  of  $\mathcal{L}$ -spaces such that*

- i) it is cofibrantly generated, with sets of generating cofibrations and acyclic cofibrations denoted by  $I$  and  $J$ , respectively*
- ii) all morphisms in  $I$  (and hence in  $J$ ) are closed embeddings of underlying spaces*
- iii) the class  $\mathcal{W}$  of weak equivalences contains all strong global equivalences (in the sense of Definition 3.1)*
- iv) the class of morphisms which are simultaneously weak equivalences and closed embeddings is closed under transfinite composition.*

*Then the category of  $*$ -modules  $\mathcal{M}_*$  admits a model structure  $(\mathcal{M}_*)_a$  satisfying the following properties:*

- a) It is cofibrantly generated, with sets of generating cofibrations and acyclic cofibrations given by  $I \boxtimes_{\mathcal{L}} *$  and  $J \boxtimes_{\mathcal{L}} *$ , respectively.
- b) The weak equivalences are precisely those morphisms of  $*$ -modules which are sent to  $\mathcal{W}$  under the forgetful functor to  $\mathcal{LU}$ .
- c) Fibrations are detected by the functor  $F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{M}_* \rightarrow (\mathcal{LU})_a$ .
- d) The adjunction

$$(\mathcal{LU})_a \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}} *} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*, -)} \end{array} (\mathcal{M}_*)_a$$

is a Quillen equivalence.

Moreover:

- e) If  $(\mathcal{LU})_a$  is right proper, then so is  $(\mathcal{M}_*)_a$ . If  $(\mathcal{LU})_a$  is a topological model category, then so is  $(\mathcal{M}_*)_a$ .
- f) If  $(\mathcal{LU})_a$  satisfies the pushout product axiom with respect to the box product, then so does  $(\mathcal{M}_*)_a$ .

Assume in addition that all elements of  $I$  are  $h$ -cofibrations in  $\mathcal{LU}$  and  $\mathcal{W}$  is a class of equivalences detected by a family of fixed point functors to spaces. Then:

- g) Both  $(\mathcal{LU})_a$  and  $(\mathcal{M}_*)_a$  are left proper.
- h) Both  $(\mathcal{LU})_a$  and  $(\mathcal{M}_*)_a$  satisfy the unit axiom and monoid axiom [15, Def. 3.3].

*Remark 5.2.* Assumption iii) can be weakened: It is enough for  $\mathcal{W}$  to contain all strong universal equivalences, since in this case the statement of Proposition 3.2 remains true, as pointed out in Remark 3.3, and the proof given below still applies.

Before turning to the proof, we record three technical, but very useful results.

**Lemma 5.3.** *Let  $\mathcal{C}, \mathcal{C}'$  be two cocomplete categories which are enriched and tensored over spaces. Let  $G: \mathcal{C} \rightarrow \mathcal{C}'$  be a continuous functor that preserves pushouts along  $h$ -cofibrations and commutes with taking tensors with the unit interval  $I$ . Then  $G$  takes  $h$ -cofibrations in  $\mathcal{C}$  to  $h$ -cofibrations in  $\mathcal{C}'$ .*

*Proof.* This is an obvious generalization of [12, Cor. A.1.11, ii)]. □

**Lemma 5.4** (Gluing lemma). *Consider the following pushout diagram in  $\mathcal{LU}$  or  $\mathcal{M}_*$ , where one of the maps  $f$  or  $g$  is an  $h$ -cofibration.*

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow g & & \downarrow h \\
Z & \xrightarrow{k} & W
\end{array}$$

If  $f$  is a global equivalence, then so is  $k$ . The statement remains true if “global equivalence” is replaced with any class of weak equivalences detected by a family of fixed point functors to spaces.

*Proof.* Colimits in  $\mathcal{M}_\ast$  are created in  $\mathcal{LU}$ ; in both settings, the statement follows from the gluing lemma for  $h$ -cofibrations and weak homotopy equivalences in spaces, since  $h$ -cofibrations are closed embeddings and taking fixed points thus commutes with forming the pushout.  $\square$

**Lemma 5.5.** *All  $\mathcal{L}$ -spaces, co- $\ast$ -modules and  $\ast$ -modules are small with respect to sequences of closed embeddings in the sense of [9, Def. 2.1.3].*

*Sketch of proof.* As the forgetful functor has both adjoints, colimits in  $\mathcal{LU}$  can be formed in  $\mathcal{U}$ . Consequently, all  $\mathcal{L}$ -spaces are small with respect to sequences of closed embeddings, and one readily verifies that the canonical map

$$j: \operatorname{colim}_{\beta < \lambda} \mathcal{LU}(A, X_\beta) \rightarrow \mathcal{LU}(A, \operatorname{colim}_{\beta < \lambda} X_\beta)$$

is not just a bijection, but even a homeomorphism, provided that  $\lambda$  is a  $\kappa$ -filtered ordinal and  $A$  is  $\kappa$ -small.

The functor  $F = F_{\boxtimes_{\mathcal{L}}}(\ast, -)$  is given explicitly as  $\mathcal{LU}(\ast, \mathcal{LU}(\mathcal{L}(2), -))$  and commutes with sequential colimits along closed embeddings by an iteration of the argument in the first part of the proof. The statement for co- $\ast$ -modules follows easily; the proof for  $\ast$ -modules also makes use of the equivalence of categories  $\mathcal{M}^\ast \simeq \mathcal{M}_\ast$ .  $\square$

In order to prove Theorem A, we construct an intermediate model structure  $(\mathcal{M}^\ast)_a$  on co-star-modules, thus exploiting the fact that, up to equivalence of categories,  $\mathcal{M}_\ast$  is a category of algebras over a well-behaved monad. This approach was used by Blumberg, Cohen and Schlichtkrull to transport their non-equivariant model structure in [1, Sect. 4.6], and goes back to [5]. Consider the following diagram:

$$(5.6) \quad \mathcal{LU} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{R} \end{array} \mathcal{LU}[\mathbb{F}] \cong \mathcal{M}^\ast \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}}^\ast} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(\ast, -)} \end{array} \mathcal{M}_\ast$$

We have seen in Proposition 2.5 that the adjunction on the right hand side is an equivalence of categories. The proof of the identification  $\mathcal{LU}[\mathbb{F}] \cong \mathcal{M}^*$  is identical with the proof of [5, Prop. II.2.7], where  $\mathbb{F}$  denotes the monad  $\mathbb{F} = RF$  associated to the adjunction on the left hand side.

There are several results which construct model structures on categories of algebras over monads, making the proof of Theorem 5.1 an almost formal issue. The following proposition is a slight variation of [15, Lemma 2.3]. Our condition (R3) is more general than that of Schwede-Shipley, but may be harder to verify in general. In the case of interest in this paper, it comes for free.

**Theorem 5.7** (Lifting of model structures). *Let  $\mathcal{C}$  be a cofibrantly generated model category and  $I$  (respectively  $J$ ) a set of generating (acyclic) cofibrations. Let  $T$  be a monad on  $\mathcal{C}$  and denote by  $I_T$  and  $J_T$  the images of  $I$  and  $J$ , respectively, under the free  $T$ -algebra functor. Assume that*

- (R1) *the domains of  $I_T$  and  $J_T$  are small relative to  $I_T$ -cell and  $J_T$ -cell, respectively*
- (R2) *every morphism in  $J_T$ -cell is sent to a weak equivalence in  $\mathcal{C}$  under the forgetful functor*
- (R3) *the category  $\mathcal{C}[T]$  of  $T$ -algebras is cocomplete.*

*Then  $\mathcal{C}[T]$  is a cofibrantly generated model category with generating sets of (acyclic) cofibrations  $I_T$  (respectively  $J_T$ ), and weak equivalences and fibrations detected by the forgetful functor to  $\mathcal{C}$ .*

**Corollary 5.8.** *Given a model category  $(\mathcal{LU})_a$  as in Theorem 5.1, the category of co- $*$ -modules admits a cofibrantly generated model structure  $(\mathcal{M}^*)_a$  with weak equivalences and fibrations detected by the forgetful functor  $R: \mathcal{M}^* \rightarrow (\mathcal{LU})_a$ . Sets of generating cofibrations and acyclic cofibrations are given by  $F_{\boxtimes_{\mathcal{C}}}(*, I)$  and  $F_{\boxtimes_{\mathcal{C}}}(*, J)$ , respectively.*

*Proof.* We verify the requirements of Theorem 5.7. All colimits exist since the forgetful functor to  $\mathcal{L}$ -spaces has a left adjoint. The smallness statement is a special case of Lemma 5.5. We now prove (R2): Let  $j: A \rightarrow B$  be a morphism in  $J$  and let the left diagram

$$\begin{array}{ccc}
 F(A) & \longrightarrow & X \\
 F(j) \downarrow & & \downarrow g \\
 F(B) & \longrightarrow & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \longrightarrow & RX \\
 j \downarrow & & \downarrow g_0 \\
 B & \longrightarrow & Y_0
 \end{array}$$

be a pushout square in  $\mathcal{M}^*$ . Then the right diagram is a “preimage square” where  $Y_0$  is taken to be the pushout of the diagram. Its image under the functor  $F$  is the pushout square

$$\begin{array}{ccc} F(A) & \longrightarrow & (FR)(X) \cong X \\ F(j) \downarrow & & \downarrow F(g_0) \\ F(B) & \longrightarrow & F(Y_0) \end{array}$$

but  $(FR)(X) \cong X$ , hence  $F(Y_0) \cong Y$  by uniqueness of the pushout, and the maps  $g$  and  $F(g_0)$  agree under this isomorphism. The map  $j$  is both an acyclic cofibration and a closed embedding, hence so is its cobase change  $g_0$  and consequently  $g \cong F(g_0)$ . Being a closed embedding and weak equivalence is stable under transfinite compositions by assumption, thus all relative  $J_T$ -cell complexes are weak equivalences.  $\square$

We are now ready to give the

*Proof of Theorem A.* The model structure  $(\mathcal{M}^*)_a$  from Corollary 5.8 transports along the equivalence of categories

$$\mathcal{M}^* \begin{array}{c} \xrightarrow{-\boxtimes_{\mathcal{L}} *} \\ \xleftarrow{F_{\boxtimes_{\mathcal{L}}}(*, -)} \end{array} \mathcal{M}_*$$

to a model structure  $(\mathcal{M}_*)_a$  with weak equivalences and fibrations detected by the composite  $R \circ F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{M}_* \rightarrow \mathcal{LU}$ , which proves c). Sets of generating (acyclic) cofibrations are given by the images of  $I$  (resp.  $J$ ) under  $F_{\boxtimes_{\mathcal{L}}}(*, -) \boxtimes_{\mathcal{L}} *: \mathcal{LU} \rightarrow \mathcal{M}_*$ , which is naturally equivalent to the functor  $(-) \boxtimes_{\mathcal{L}} *$  by Proposition 2.5, thus proving a). Hypothesis iii) and Proposition 3.2 imply that  $F_{\boxtimes_{\mathcal{L}}}(*, -)$  preserves and reflects the weak equivalences  $\mathcal{W}$ ; now b) follows immediately. We constructed  $(\mathcal{M}_*)_a$  in a way such that both adjoint pairs in Diagram 5.6 are Quillen adjunctions. The unit  $\bar{\lambda}: X \rightarrow RF(X) = F_{\boxtimes_{\mathcal{L}}}(*, X)$  is a strong global equivalence by Proposition 3.2, hence an element of  $\mathcal{W}$ . Let  $X \in \mathcal{LU}, M \in \mathcal{M}_*$  (without any (co-)fibrancy assumptions), then a map  $X \rightarrow F_{\boxtimes_{\mathcal{L}}}(*, M)$  is in  $\mathcal{W}$  if and only if its factorization  $F_{\boxtimes_{\mathcal{L}}}(*, X \boxtimes_{\mathcal{L}} *) \rightarrow F_{\boxtimes_{\mathcal{L}}}(*, M)$  is. Since b) holds, the left pair is a Quillen equivalence. So is the right pair by construction, and hence the composite, i.e., statement d) holds.

Now we proof the enhancements e) through h):

*Ad e):* Assume that  $(\mathcal{LU})_a$  is right proper. Then so is  $(\mathcal{M}_*)_a$  since the right adjoint  $R \circ F_{\boxtimes_{\mathcal{L}}}(*, -): \mathcal{M}_* \rightarrow \mathcal{LU}$  preserves pullbacks, and preserves and

reflects weak equivalences and fibrations.

Now assume that  $(\mathcal{LU})_a$  is topological. Let  $f: X \rightarrow Y$  be a generating cofibration for  $(\mathcal{LU})_a$  and  $i: A \rightarrow B$  any cofibration in  $\mathcal{U}$ . By assumption, the pushout product

$$f \square i: P = Y \times A \cup_{X \times A} X \times B \rightarrow Y \times B$$

is again a cofibration in  $(\mathcal{LU})_a$ . The map  $f \boxtimes_{\mathcal{L}} *$  is a generating cofibration in  $(\mathcal{M}_*)_a$  whose pushout product with  $i$  is isomorphic to

$$(f \square i) \boxtimes_{\mathcal{L}} *: P \boxtimes_{\mathcal{L}} * \rightarrow (Y \times B) \boxtimes_{\mathcal{L}} *.$$

As  $- \boxtimes_{\mathcal{L}} *: (\mathcal{LU})_a \rightarrow (\mathcal{M}_*)_a$  is a left Quillen functor, this map is a cofibration in  $\mathcal{M}_*$ . If  $f$  is a generating acyclic cofibration or  $i$  any acyclic cofibration, then  $f \square i$  is an acyclic cofibration in  $\mathcal{LU}$ , hence so is  $(f \boxtimes_{\mathcal{L}} *) \square i \cong (f \square i) \boxtimes_{\mathcal{L}} *$  in  $\mathcal{M}_*$ .

*Ad f):* There are natural isomorphisms

$$(X \boxtimes_{\mathcal{L}} *) \boxtimes_{\mathcal{L}} (X' \boxtimes_{\mathcal{L}} *) \cong (X \boxtimes_{\mathcal{L}} X') \boxtimes_{\mathcal{L}} *$$

for all  $\mathcal{L}$ -spaces  $X$  and  $X'$ . Similar reasoning as in the proof of g) then shows that for two generating cofibrations  $f: A \rightarrow B$  and  $f': A' \rightarrow B'$  for  $(\mathcal{LU})_a$ , the pushout product of  $f \boxtimes_{\mathcal{L}} *$  and  $f' \boxtimes_{\mathcal{L}} *$  is isomorphic to  $(f \square f') \boxtimes_{\mathcal{L}} *$ , hence is a cofibration in  $\mathcal{M}_*$ , and acyclic if  $f$  or  $f'$  is a generating acyclic cofibration.

*Ad g):* Left properness follows immediately from Lemma 5.4.

*Ad h):* The box product is weakly equivalent to the categorical product by Proposition 3.2 and the assumption that any strong global equivalence is a weak equivalence in  $(\mathcal{LU})_a$ . As the weak equivalences are detected by fixed point functors, the functor  $(-) \boxtimes_{\mathcal{L}} Z$  preserves weak equivalences, where  $Z \in \mathcal{LU}$  is any  $\mathcal{L}$ -space. The unit axiom follows immediately.

Let  $\mathcal{A}$  denote the class of morphisms  $j \boxtimes_{\mathcal{L}} Z$  where  $j$  is an acyclic cofibration and  $Z \in \mathcal{LU}$  is arbitrary. All cofibrations in  $(\mathcal{LU})_a$  are  $h$ -cofibrations. As just observed, the functor  $(-) \boxtimes_{\mathcal{L}} Z$  preserves weak equivalences. Because of Lemma 5.3, it always preserves  $h$ -cofibrations, too. Moreover, the class of weak equivalences which are  $h$ -cofibrations is stable under cobase changes (by Lemma 5.4), composition, and retracts. Thus, all relative  $\mathcal{A}$ -complexes are weak equivalences.

The same proof applies to  $(\mathcal{M}_*)_a$ . □



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